Available online at www.sciencedirect.com
SCIENCE DIRECT®

J. Differential Equations 197 (2004) 219–246

Journal of Differential Equations

http://www.elsevier.com/locate/jde

On perturbations of delay-differential equations with periodic orbits

Jack K. Hale[a] and Marion Weedermann[b,*]

[a] School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA
[b] Natural and Applied Sciences, University of Wisconsin Green Bay, Green Bay, WI 54311, USA

Received October 5, 2001; revised April 5, 2002

Abstract

Equations of retarded type and simple neutral-type equations are considered. The study concerns both autonomous and non-autonomous perturbations of an autonomous equation which possesses a non-trivial periodic orbit. The main tool is a local coordinate system around the periodic orbit which is obtained from the phase space decomposition via Floquet multipliers. Under the assumption that the perturbation function is Lipschitz the existence of an integral manifold with periodic structure for the system in the new coordinates is shown. This implies that, under autonomous perturbations, periodic orbits are continued. Furthermore, we give a description of the flow on the center manifold of the periodic orbit.

Keywords: Functional differential equation; Periodic orbit; Floquet multiplier; Exponential dichotomy; Integral manifold

1. Introduction

For a periodic orbit of an autonomous ordinary differential equation in $\mathbb{R}^n$, it is known that there is a moving coordinate system consisting of an "angular" variable and an $(n-1)$-dimensional "normal" coordinate [7]. An analysis of the equations in the new coordinates permits one to answer stability questions as well as questions regarding the existence of integral manifolds when the vector field is subjected to autonomous as well as non-autonomous perturbation. Henry [10] introduced a

*Corresponding author. 7239 B Franklin Street, Forest Park, IL 60130, USA.
E-mail addresses: jack.hale@math.gatech.edu (J.K. Hale), weedermann@uwgb.edu (M. Weedermann).

doi:10.1016/S0022-0396(02)00063-3

coordinate system around an invariant manifold of a parabolic partial differential equation. His construction made use of the idea of trivial stability of the tangent bundle over the invariant manifold. When applied to hyperbolic periodic orbits, that coordinate system allows one to obtain results for a wide range of perturbations including periodic and almost periodic perturbations. He also was able to extend those results to parameterized families of periodic orbits. The work presented here was in many ways inspired by Henry's result. The construction of the coordinate system, however, is based on a different idea.

For retarded functional differential equations (FDEs) the problem of continuation of a periodic orbit under autonomous perturbations has been studied in [9, pp. 325–328]. The results states that if the equation $\dot{x} = f(x_t)$ has a non-degenerate periodic orbit of minimal period ω, then the perturbed equation $\dot{x}(t) = f(x_t) + g(x_t, \lambda)$, where $g(x_t, 0) = 0$, has a solution for small λ which has the parametric representation $x = u(s)$, the function u is ω-periodic in s, and s is a function of the time t. The proof involves a change of the phase space and cannot be generalized to answer the same question for non-autonomous perturbations. A different approach which does not require the change of the phase space was taken by Stokes [14]. Stokes introduced a local coordinate system around a periodic orbit of a retarded FDE in such a way as to be able to determine many properties of the solution in a neighborhood of the periodic orbit. That coordinate system cannot be considered a natural generalization of the finite dimensional case because it consisted of three parts; an angular coordinate, a normal coordinate and a tangential coordinate. Due to the third coordinate, Stokes was not able to apply his results to the case of non-degenerate periodic orbits and obtain a center manifold for the orbit.

The standard method for studying the behavior near a periodic orbit under autonomous perturbations is to use the Poincaré map. If the Poincaré map is differentiable, the problem can be reduced to the problem of studying the behavior near the fixed point of that map. However, this method is restricted to autonomous equations. Also, for neutral FDE, even in the autonomous case, the Poincaré map is not necessarily C^1.

In this paper, we introduce a moving coordinate system for retarded FDE as well as for an important class of neutral FDE. This coordinate system makes extensive use of the decomposition theory with respect to the Floquet multipliers with the "normal" coordinate being closely related to a natural bilinear form involving the adjoint equation. We support our definition of a moving coordinate system by showing that exponential dichotomy is preserved and we obtain the expected relationships between the Floquet multipliers of the original and the new equation.

The paper is organized as follows. In Section 2, we give some basic results for FDEs and discuss the decomposition of the phase space. In Section 3, we define a moving coordinate system around a non-degenerate periodic orbit and relate the Floquet multipliers in the 'normal' direction to the Floquet multipliers of the periodic orbit. In Section 4, we show that the new equations possess exponential dichotomy. We present a general theorem which ensures the existence of an integral manifold. The proof of this theorem is similar to the one given by Henry [10], see also [13]. In Section 5, we turn to the study of autonomous or non-autonomous perturbations of an autonomous neutral FDE under the assumption that it possesses a non-trivial periodic solution. We first consider the case when the periodic orbit is hyperbolic to obtain the existence of a hyperbolic invariant manifold. Next, we turn to the study of the non-degenerate case and prove a general center manifold result. If the perturbation is autonomous, we give a result on the continuation of the periodic orbit which is new for neutral FDE. Finally, we restrict our attention to the study of autonomous equations and show that it is possible to give a description of the flow on the center manifold.

2. Notation and some basic results for FDE

Let $r > 0$ be a fixed real number, the delay. Define $C = C([-r, 0], \mathbb{R}^n)$ to be the space of continuous functions mapping the interval $[-r, 0]$ into $\mathbb{R}^n$. Together with the supremum norm, $|\phi|_C = \sup\{|\phi(\theta)| : -r \leqslant \theta \leqslant 0\}$ for $\phi \in C$, C is a Banach space. In this paper, for a given continuous function $x : \mathbb{R} \to \mathbb{R}^n$, we will denote by x_t a function in C with the property that $x_t(\theta) = x(t + \theta)$, for $-r \leqslant \theta \leqslant 0$.

If $\mu(\theta)$, $\theta \in [-r, 0]$ is an $n \times n$ matrix function of bounded variation, we say that it is *non-atomic at* β, if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that the total variation $Var_{[-\delta+\beta, \delta+\beta]}\mu(\cdot) < \varepsilon$.

In this paper, we consider neutral functional differential equations of the form

$$\frac{d}{dt} Dx_t = L(t)x_t + f(t, x_t), \tag{2.1}$$

where $f : \mathbb{R} \times C \to \mathbb{R}^n$ is continuous. The linear, continuous operator $D : C \to \mathbb{R}^n$ is of the form

$$D\phi = \phi(0) - \int_{-r}^{0} d[\mu(\theta)]\phi(\theta), \tag{2.2}$$

where μ is an $n \times n$ matrix of bounded variation which is non-atomic at zero, continuous from the left on $(-r, 0)$ and $\mu(0) = 0$. Furthermore, we assume that the operator D is stable [3], that is, the zero solution of the difference equation $Dy_t = 0$ with $y_0 \in \{\phi \in C : D\phi = 0\}$ is uniformly asymptotically stable. The linear mapping $L(t) : C \to \mathbb{R}^n$ is given by

$$L(t)\phi = \int_{-r}^{0} d_\theta[\eta(t, \theta)]\phi(\theta), \tag{2.3}$$

where $\eta(t, \theta)$ is a measurable $n \times n$ matrix function, continuous from the left in θ, has bounded variation in θ on $[-r, 0]$ for each t and is normalized so that

$$\eta(t, \theta) = 0 \quad \text{for } \theta \geqslant 0, \qquad \eta(t, \theta) = \eta(t, -r) \quad \text{for } \theta \leqslant -r.$$

Also, we assume that there is an $m \in \mathfrak{L}_1^{\text{loc}}((-\infty, \infty), \mathbb{R})$ such that $Var_{[-r,0]}\eta(t, \cdot) \leqslant m(t)$. It is easily seen that the last condition implies that $|L(t)\phi|_{\mathbb{R}^n} \leqslant m(t)|\phi|_C$.

If $D\phi = \phi(0)$, Eq. (2.1) reduces to a retarded FDE

$$\dot{x}(t) = L(t)x_t + f(t, x_t). \tag{2.4}$$

2.1. Linear FDE

Let $x_t(\cdot\,; \phi)$ be the unique solution of the linear homogeneous equation

$$\frac{d}{dt} Dx_t = 0, \tag{2.5}$$

with $x_0(\cdot\,; \phi) = \phi$. The solution operator $T(t) : C \to C$, $t \geqslant 0$, is defined by the relation

$$T(t)\phi \stackrel{\text{def}}{=} x_t(\cdot\,; \phi).$$

It has been shown [9] that $T(t)$ is a C_0-semigroup. The infinitesimal generator A associated with $T(t)$ defined by

$$A\phi = \lim_{h \to 0} \frac{1}{h} [T(t+h)\phi - T(t)\phi], \quad \phi \in \mathfrak{D}(A),$$

with $\mathfrak{D}(A)$ being the set of all $\phi \in C$ for which the limit exists, satisfies the property that

$$\frac{d}{dt} T(t)\phi = A \; T(t)\phi = T(t)A\phi \qquad \text{for every } \phi \in \mathfrak{D}(A).$$

By direct computation one can show that, for Eq. (2.5), the infinitesimal generator is given by

$$\mathfrak{D}(A) = \{\phi \in C^1 \mid D\phi' = 0\} \quad \text{and} \quad A\phi = \phi', \text{ where } \phi' = \frac{d\phi}{d\theta}.$$

The operator A can be extended to C^1 as follows. Denote by BC the space of functions which are uniformly continuous on $[-r, 0)$ and have a finite jump discontinuity at 0. Functions ψ in BC can be represented as $\psi = \phi + X_0\alpha$, where $\phi \in C$, $\alpha \in \mathbb{R}^n$, and

$$X_0(\theta) = \begin{cases} 0, & -r \leqslant \theta < 0, \\ I_{n \times n}, & \theta = 0. \end{cases}$$

$(BC; ||\cdot||_{BC})$ is a Banach space, with the norm $||\psi||_{BC} \stackrel{\text{def}}{=} |\phi|_C + |\alpha|_{R^n}$. Define the operator A_0 by

$$A_0\phi \coloneqq \phi' - X_0[D\phi'],$$

where $\phi' = \frac{d\phi}{d\theta}$. The domain of A_0 is C^1. Note that $C^1 \subset BC$ and $A_0|_{C^1} = A$. If we consider the equation

$$\frac{d}{dt}Dx_t = L(t)x_t + f(t, x_t) \tag{2.6}$$

as a perturbation of (2.5), Eq. (2.6) can be written as an abstract ordinary differential equation on BC

$$\frac{d}{dt}x_t = A_0x_t + X_0L(t)x_t + X_0f(t, x_t). \tag{2.7}$$

Representations of this type have been applied successfully to integral averaging [2] as well as in the development of a normal form theory on invariant manifolds near equilibrium points for retarded [5,6] and for neutral FDE [15].

2.2. Floquet multipliers

Suppose $f \in C^2(C, \mathbb{R}^n)$, and assume that the autonomous equation

$$\frac{d}{dt}Dx_t = f(x_t) \tag{2.8}$$

has a non-trivial periodic solution $p(t)$ of minimal period ω, that is, $p_t \neq 0$ for all t, and $p_t \neq p_0$, for $0 < t < \omega$. Corollaries of results in [3,8] guarantee that if D is stable and f has continuous, bounded derivatives of order $k \geqslant 0$, then p_t has continuous, bounded derivatives of order k [9]. Thus, the linear variational equation around p given by

$$\frac{d}{dt}Dx_t = L(t)x_t, \tag{2.9}$$

where $L(t) = f'(p_t)$, has the non-trivial ω-periodic solution $\dot{p}$.

It is easy to see that $L(t) \in \mathfrak{L}(C, \mathbb{R}^n)$ and that $L(\cdot)$ is ω-periodic in t. $L(t)$ can be written in the form

$$L(t)\phi = \int_{-r}^{0} d_\theta[\eta(t, \theta)]\phi(\theta),$$

with η as in (2.3). Furthermore, there exists a constant m_1 such that, for all $t \in \mathbb{R}$, $\int_t^{t+r} m(\theta)\, d\theta \leqslant m_1$.

If $x(t; s, \phi)$ is the solution of (2.9) with $x_s(\cdot\,; \phi) = \phi$, let $T(t, s)\phi = x_t(s, \phi)$, $t \geqslant s$ and $\phi \in C$, be the solution operator corresponding to Eq. (2.9). Let $U(s) : C \to C$ be defined by $U(s) = T(s + \omega, s)$, $s \in \mathbb{R}$.

Consider the point spectrum $\sigma_P(U(s))$ of $U(s)$ and observe that $\mu \in \sigma_P(U(s_1))$ implies $\mu \in \sigma_P(U(s_2))$. Since the point spectrum is independent of the starting time it is justified to make the following definition.

Definition 1. An element $\mu \neq 0$ in $\sigma_P(U) \stackrel{\text{def}}{=} \sigma_P(U(0))$ is called a Floquet multiplier of Eq. (2.9).

For any Floquet multiplier μ of (2.9), there are two closed subspaces $E_\mu(s)$ and $Q_\mu(s)$, $s \in \mathbb{R}$, such that

(i) $E_\mu(s)$ is finite dimensional,
(ii) $E_\mu(s) \oplus Q_\mu(s) = C$,
(iii) $T(t,s)E_\mu(s) = E_\mu(t)$ for $t \in \mathbb{R}$ and $T(t,s)Q_\mu(s) \subseteq Q_\mu(t)$ for $t \geqslant s$.
(iv) $\sigma(U(s)|E_\mu(s)) = \{\mu\}$,
$\sigma(U(s)|Q_\mu(s)) = \sigma(U(s)) \setminus \{\mu\}$.
(v) The sets $E_\mu(s)$ and $E_\mu(t)$ are diffeomorphic,
the sets $Q_\mu(s)$ and $Q_\mu(t)$ are homeomorphic.

The dimension of E_μ is called the *multiplicity* of the Floquet multiplier μ. Since $\dot{p}_t \neq 0$ is a solution of (2.9), $\mu = 1$ is always a Floquet multiplier of (2.9).

Definition 2. The *Floquet multipliers of a periodic orbit Γ* are the Floquet multipliers of the linear variational equation (2.9) except that 1 is not a multiplier of Γ if 1 is a simple multiplier of (2.9).

Definition 3. A periodic orbit Γ of (2.8) is *non-degenerate* if 1 is not a Floquet multiplier of Γ. It is said to be *hyperbolic* if each Floquet multiplier of Γ has modulus different from 1.

If μ is a Floquet multiplier of (2.9), then there exist subspaces $E_\mu(s)$ and $Q_\mu(s)$ such that $C = E_\mu(s) \oplus Q_\mu(s)$. It has been shown that there exists a d_μ such that $E_\mu(s) = \mathfrak{Ker}\,((\mu I - U(s))^{d_\mu})$. If $\Phi_\mu(s)$ is a basis for $E_\mu(s)$, then $T(t,s)\Phi_\mu(s) = \Phi_\mu(t)$ is a basis for $E_\mu(t)$. Furthermore, $U(s)\Phi_\mu(s) = \Phi_\mu(s)M_\mu(s)$ where $M_\mu(s)$ is a $d_\mu \times d_\mu$ matrix with $\sigma(M_\mu(s)) = \{\mu\}$ for all $s \in (-\infty, \infty)$.

Define $U = T(\omega, 0)\Phi(0)$ and let B_μ be the $d_\mu \times d_\mu$ matrix defined by $B_\mu = \omega^{-1} \log M_\mu(0)$. It is easy to see that $\sigma(B_\mu) = \{\zeta\}$, where ζ is such that $e^{\zeta\omega} = \mu$. The value ζ is commonly referred to as the *characteristic exponent* corresponding to μ. Note that, unless $\mu = 1$, $\zeta \neq 0$. Also, note that if $\mu = -1$, then the matrix B could be complex. To avoid this difficulty, one could double the period. As a result, one would obtain one additional Floquet multiplier 1 and the matrix B would be real. Define the vector $P(t) = T(t,0)e^{-B_\mu t}$. Then, for $t \geqslant 0$, $P(t) = P(t+\omega)$ and $U\Phi(0) = \Phi(0)e^{B\omega}$. Let $C^* = C([0,r], \mathbb{R}^{n\star})$, where $\mathbb{R}^{n\star}$ denotes the space of row-vectors in $\mathbb{R}^n$.

We define a bilinear form $\langle \cdot , \cdot \rangle_t$ on $C \times C^*$ as follows:

$$\langle \psi, \phi \rangle_t = \psi(0)\phi(0) - \int_{-r}^{0} \int^{\theta}{}_0 \psi'(\tau - \theta)\, d\mu(\theta)\ \phi(\tau)\, d\tau$$
$$- \int_{-r}^{0} \int_{0}^{\theta} \psi(\tau - \theta)\, d_\theta[\eta(t + \theta - \tau, \theta)]\ \phi(\tau)\, d\tau$$

for $\phi \in C$ and $\psi \in C^*$. Corresponding to the basis $\Phi(s)$ of $E_\mu(s)$ there exists a set of continuous functions $\Psi(s) \subset C^*$ such that $\langle \Psi_\mu(s), \Phi_\mu(s) \rangle_s = I_{d_\mu \times d_\mu}$ and

$$E_\mu(s) = \{\phi : \phi = \Phi_\mu(s) \langle \Psi_\mu(s), \phi \rangle_s\},$$

$$Q_\mu(s) = \{\phi : \langle \Psi_\mu(s), \phi \rangle_s = 0\}.$$

We refer to $\Psi_\mu(s)$ as the *dual basis* of $\Phi_\mu(s)$. Using the above properties, one can verify that $\Psi_\mu(s) = e^{-B_\mu s} P^T(s)$.

When deriving Eq. (2.7) it was necessary to extend the phase space C to BC. We therefore need a decomposition of BC. The projection corresponding to the decomposition of C, $\pi_\mu(s) : C \to E_\mu(s)$ is given by $\pi_\mu(s)\phi = \Phi_\mu(s)\langle \Psi_\mu(s), \phi \rangle_s$. This projection can be extended to $\tilde{\pi}_\mu(s) : BC \to E_\mu(s)$ by

$$\tilde{\pi}_\mu(s)(\phi + X_0\alpha) = \Phi_\mu(s)[\langle \Psi_\mu(s), \phi \rangle_s + \Psi_\mu(s)(0)\alpha].$$

It is clear that $Q_\mu(s) \subset \mathfrak{Ker}\, \tilde{\pi}_\mu(s)$. More precisely, $\mathfrak{Ker}\, \tilde{\pi}_\mu(s) \simeq Q(s) \oplus [X_0]$. Using integration by parts and adjoint theory, one can show that $A_s = A_0 + X_0 f'(p_s)$ and $\tilde{\pi}_\mu(s)$ commute on C^1.

Suppose that the equation

$$\frac{d}{dt} D x_t = f(x_t)$$

has a non-trivial periodic orbit $\Gamma = \{p_t : 0 \leqslant t \leqslant \omega\}$. The equation can be rewritten as an abstract equation

$$\dot{x}_t = A_0 x_t + X_0[f'(p_t)x_t] + X_0[f(x_t) - f'(p_t)x_t]. \tag{2.10}$$

Assuming that x_t satisfies Eq. (2.10) and writing

$$x_t = P_\mu(t)y(t) + q(t), \quad y \in \mathbb{R}^{d_\mu},$$

$$q(t) \in \mathfrak{Ker}\, \tilde{\pi}_\mu(t) \cap \mathfrak{D}(A_0) = Q_\mu(t) \cap C^1$$

yields

$$\dot{y}(t) = B_\mu y(t) + \Psi_\mu(t)(0)[f(P_\mu(t)y(t) + q(t)) - f'(p_t)(P_\mu(t)y(t) + q(t))]$$

$$\begin{aligned}\dot{q}(t) = &A_0 q(t) + X_0 f'(p_t)q(t)\\ &+ (I - \tilde{\pi}_\mu(t))X_0[f(P_\mu(t)y(t) + q(t)) - f'(p_t)(P_\mu(t)y(t) + q(t))].\end{aligned}$$

This decomposition can be extended to a finite set $\Lambda = \{\mu_1, \mu_2, \dots, \mu_k\}$ of Floquet multipliers. We can decompose C into

$$C = E_{\mu_1}(t) \oplus E_{\mu_2}(t) \oplus \cdots \oplus E_{\mu_k}(t) \oplus Q_\Lambda(t),$$

where $Q_\Lambda(t)$ is such that $T(t,s)Q_\Lambda(s) \subseteq Q_\Lambda(t)$. Let B_Λ be the $d_\Lambda \times d_\Lambda$, ($d_\Lambda = \sum_{i=1}^k d_{\mu_i}$), matrix given by

$$B_\Lambda = diag(B_{\mu_1}, \dots, B_{\mu_k}),$$

let Φ_Λ and Ψ_Λ be the corresponding collection of bases and adjoint bases, respectively. Note that the spectrum of the matrix B_Λ does not consist of the Floquet multipliers in Λ but of corresponding characteristic exponents. Define $P_\Lambda(t) = \Phi_\Lambda(t)e^{-B_\Lambda t}$. Then, if we set $x_t = \Phi_{\mu_1}(t)y_1(t) + \cdots + \Phi_{\mu_k}(t)y_k(t) + q(t)$, $q(t) \in Q_\Lambda(t) \cap C^1$, we obtain ($i = 1, \dots, k$)

$$\dot{y}_i(t) = B_{\mu_i} y_i(t) + \Psi_{\mu_i}(t)(0)[f(P_\Lambda(t)y(t) + q(t)) - f'(p_t)(P_\Lambda(t)y(t) + q(t))]$$

$$\begin{aligned}\dot{q}(t) = &A_0 q(t) + X_0 f'(p_t)q(t)\\ &+ (I - \tilde{\pi}_\Lambda(t))X_0[f(P_\Lambda(t)y(t) + q(t)) - f'(p_t)(P_\Lambda(t)y(t) + q(t))].\end{aligned}$$

3. Local coordinates near a periodic orbit

We will use the decomposition with respect to Floquet multipliers to introduce a local moving coordinate system around the periodic orbit. For the remainder of the paper we assume that the periodic orbit is non-degenerate.

3.1. Moving coordinates

Suppose $\Gamma = \{p_t : 0 \leqslant t \leqslant \omega\}$ is a non-trivial periodic orbit of (2.8). Consider a phase space decomposition with respect to the Floquet multiplier $\mu = 1$. If Γ is non-degenerate, i.e., $\mu = 1$ is a simple Floquet multiplier of (2.9), then a decomposition of C is given by

$$C = [\dot{p}_s] \oplus Q_1(s),$$

where $Q_1(s)$ is such that $T(t,s)Q_1(s) \subseteq Q_1(t)$. It is possible to define this decomposition when 1 is not a simple Floquet multiplier of (2.9). In that case, $E_1(s)$ has higher dimension with a basis $\Phi_1(s) = \{\dot{p}_s, \phi_2, \dots, \phi_{d_1}\}$. Since the dual basis is such that $\langle \Psi_1(s), \Phi_1(s) \rangle_s = Id$, for all s, it is still possible to decompose the space C into $C = [\dot{p}_s] \oplus Q(s)$, where *codim* $Q(s) = 1$. On the other hand, we may lose the invariance property $T(t,s)Q(s) \subseteq Q(t)$ of the complementary subspace. If $U(0)\Phi_1(0) = \Phi_1(0)e^{-B_1\omega}$ and

$$\text{(B)} \qquad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & \hat{B}_1 \end{pmatrix},$$

then this invariance property will be preserved and the analysis which follows will be valid. We keep the hypothesis of non-degeneracy of Γ to minimize notation. In the case when the periodic orbit is such that condition (B) is satisfied, the decomposition with respect to the Floquet multiplier 1 provides us with an "orthogonal complement" of $\dot{p}_s$.

Definition 4 (Local coordinate system along Γ). By a local coordinate system along Γ, we mean that there exists a co-dimension 1 subspace M_0 of C such that, for all $s \in [0, \omega]$, there exists a map L_s from M_0 to $Q(s)$ which is an isomorphism onto its range and there is a neighborhood V of Γ such that for every $\phi \in V$, ϕ has the decomposition $\phi = p_s + L_s w$, where $w \in M_0$.

If Γ is non-degenerate, then $E_1(s) = [\dot{p}_s]$, in the more general case under hypothesis (B), $E_1(s) = [\dot{p}_s] \oplus E_1(s)$. All of the following results are proved for the case when Γ is non-degenerate but can be generalized for non-hyperbolic periodic orbits that satisfy condition (B). Throughout this paper, we will denote the solution of the adjoint equation of (2.9) corresponding to the periodic solution $\dot{p}_s$ by q^s. We will refer to q^s as the *dual solution* corresponding to $\dot{p}_s$. From the previous section it follows that $\langle q^s, \dot{p}_s \rangle_s = 1$. The function q^s possesses the same smoothness properties as $\dot{p}_s$.

Every element in C has a unique representation of the form $\phi = \langle q^s, \phi \rangle_s \dot{p}_s + \phi^{Q_1(s)}$. For an element w of $M_0 \subset C$ we obtain $w = \langle q^s, w \rangle_s \dot{p}_s + L_s w$, where L_s is defined by

$$L_s w = w - \langle q^s, w \rangle_s \dot{p}_s. \tag{3.1}$$

Proposition 1. *Suppose $f \in C^1(C, \mathbb{R}^n)$.*

(i) *For every $v^t \in C^{\star 1} = C^1(C, \mathbb{R}^{n\star})$, $\phi \in C$,*

$$|\langle v^t, \phi \rangle_t| \leqslant a\, k_{v,v'} |\phi|.$$

(ii) *For every $\phi \in C^1 = C^1(C, \mathbb{R}^n)$,*

$$\frac{d}{dt} \langle q^t, \phi \rangle_t = q(t)[D\phi' - f'(p_t)\phi] - \langle q^t, \phi' \rangle_t.$$

(iii) *For any* $\phi \in C$, $|\alpha_s - \alpha_t| \leqslant c|s-t|\,|\phi|$, *where* $\alpha_\tau = \langle q^\tau, \phi \rangle_\tau$.
(iv) *For any* $\phi_1, \phi_2 \in C$, $|\langle q^s, \phi_1 \rangle_s - \langle q^t, \phi_2 \rangle_t| \leqslant c_1|s-t| + c_2|\phi_1 - \phi_2|$.
(v) $L_s : M_0 \cap C^1 \to Q_1(s)$ *is uniformly bounded, and continuously differentiable.*

Proof. (i) Using the representation of the bilinear form given in Section 2.3, one sees that from the bounded variation of μ and η together with the compactness of Γ, $\{q^t\}$ and $\{q^t\}$, we obtain the desired result.

(ii) By direct evaluation.

(iii)
$$
\begin{aligned}
&|\langle q^t, \phi \rangle_t - \langle q^s, \phi \rangle_s| \\
&\leqslant |q(t) - q(s)|\,|\phi(0)| \\
&\quad + \int_0^r |q^t(\alpha)| \left[\int_{-r}^0 d_\theta |\eta(t, \theta - \alpha) - \eta(s, \theta - \alpha)|\,|\phi(\theta)| \right] d\alpha \\
&\quad + \int_0^r |q^t(\alpha) - q^s(\alpha)| \left[\int_{-r}^0 d_\theta |\eta(s, \theta - \alpha)|\,|\phi(\theta)| \right] d\alpha \\
&\leqslant l_q|t-s|\,|\phi| + k_q l_{f'}|t-s|\,|\phi| + l_q k_{f'}|t-s|\,|\phi| \\
&\leqslant c|t-s|\,|\phi|.
\end{aligned}
$$

(iv)
$$
\begin{aligned}
&|\langle y^s, \phi_1 \rangle_s - \langle y^t, \phi_2 \rangle_t| \\
&\leqslant |\langle y^s, \phi_1 \rangle_s - \langle y^t, \phi_1 \rangle_t| + |\langle y^t, \phi_1 \rangle_t - \langle y^t, \psi_2 \rangle_t| \\
&\leqslant c|s-t||\phi_1| + |\langle y^t, \phi_1 - \psi_2 \rangle_t| \\
&\leqslant ck_{\phi_1}|s-t| + ak_y|\phi_1 - \psi_2|.
\end{aligned}
$$

(v) Representation (3.1) implies that L_s is continuous. This, together with the compactness of $\dot{\Gamma} = \{\dot{p}_s \mid 0 \leqslant s \leqslant \omega\}$ and (i), gives the uniform boundedness of L_s. Using representation (3.1) and (ii) one can first show that L_s is differentiable. The derivative is found to be

$$
\partial_s L_s w = -\frac{d}{ds} \langle q^s, w \rangle_s \dot{p}_s - \langle q^s, w \rangle_s \ddot{p}_s.
$$

Since $f \in C^1(C, \mathbb{R}^n)$ implies that $\ddot{p}^s \in C$, the derivative of L_s is continuous. $\square$

3.2. The new equations

Let M_0 be a co-dimension 1 subspace of C, for instance, $M_0 = Q_1(0)$. In that case, $L_1 = Id$.

Lemma 1 (Local coordinate system about Γ). *Suppose* $\Gamma = \{p_s \mid 0 \leqslant s \leqslant \omega\}$ *is a non-trivial periodic orbit of* (2.8) *of minimal period* ω. *Then there exists a neighborhood* V *of* Γ *and an* $\varepsilon > 0$ *such that, for every* $\phi \in V$, *there exists a unique pair* $(s, w) \in [0, \omega) \times M_{0,\varepsilon} = \{w \in M_0 : |w|_C < \varepsilon\}$ *such that* $\phi = p_s + L_s w$.

Proof. Let $F(\phi, s, w) = p_s + L_s w - \phi$. Then $\partial_s F = \dot{p}_s + \partial_s L_s w$, which is non-zero at $w = 0$. The partial derivative $\partial_s F$ is continuous. Therefore, by the implicit function theorem there exists a $\delta > 0$ such that $\phi = p_s + L_s w$ is well defined for $|L_s w| < \delta$. Let $\varepsilon = \delta / k_L$, where $k_L = \max\{||L_s||_{\mathfrak{L}(M_0, Q_1(s))} : s \in [0, \omega)\}$. The neighborhood V of Γ is given by $V = \{\phi \in C \mid \sup_{s \in [0,\omega)} |\phi - p_s| < \delta\}$. $\square$

Suppose $x_t \in V$ is a solution of Eq. (2.8) or, in abstract form, a solution of

$$\frac{d}{dt} x_t = A_0 x_t + X_0 f(x_t). \tag{3.2}$$

Let

$$x_t = p_{s(t)} + L_{s(t)} w(t), \tag{3.3}$$

with $w(t) \in M_{0,\varepsilon}$ and $s : [0, \infty) \to \mathbb{R}$ differentiable. With this representation of x_t, (3.2) becomes

$$\begin{aligned}
&[\dot{p}_{s(t)} + \partial_s L_{s(t)} w(t)] \dot{s}(t) + L_{s(t)} \dot{w}(t) \\
&\quad = A_0 (p_{s(t)} + L_{s(t)} w(t)) + X_0 f(p_{s(t)} + L_{s(t)} w(t)),
\end{aligned}$$

Applying $\langle q^s, \cdot \rangle_s$ gives

$$\dot{s} = 1 + f_1(s, w),$$

where

$$\begin{aligned}
f_1(s, w) = &[1 + \langle q^s, \partial_s L_s w \rangle_s]^{-1} \\
&\times (-\langle q^s, \partial_s L_s w \rangle_s + q^s(0)[f(p_s + L_s w) - f(p_s)]).
\end{aligned}$$

The inverse exists if we choose ε sufficiently small. On the other hand, projecting onto $Q(s)$ results in

$$\dot{w} = A(s) w + f_2(s, w),$$

where

$$\begin{aligned}
A(s) &= L_s^{-1} (-\partial_s L_s + A_0 L_s + X_0 f'(p_s) L_s), \\
f_2(s, w) &= L_s^{-1} (-\partial_s L_s w f_1(s, w) + X_0 [f(p_s + L_s w) - f(p_s) - f'(p_s) L_s w]).
\end{aligned}$$

It is necessary to extend L_s to $BM = M_0 \oplus [X_0]$. This can be done by defining $L_s X_0 \alpha = X_0 \alpha - q^s(0)\alpha \dot{p}_s$. In that way, if $\psi \in Q(s) \cap \Re(L_s)$, we still have $L_s^{-1}\psi \in M_0$. With this definition we have that for $w \in M_0 \cap C^1$

$$L_s A_0 w = w' - X_0 D w' - \langle q^s, w' \rangle \dot{p}_s + q^s(0) D w' \dot{p}_s,$$

$$A_0 L_s w = w' - X_0 D w' - \langle q^s, w \rangle [\dot{p}_s' - X_0 \dot{p}_s'(0)],$$

where $' = d/d\theta$. Using that

$$\dot{p}_s' = \frac{d}{d\theta}\dot{p}_s = A_0 \dot{p}_s + X_0 D \dot{p}_s'$$

together with the fact that $\dot{p}_s$ solves Eq. (2.9) we find that

$$A_0 L_s w = L_s A_0 w + \langle q^s, w' \rangle \dot{p}_s - q^s(0) D w' \dot{p}_s - \langle q^s, w \rangle (\ddot{p}_s - X_0 f'(p_s)\dot{p}_s).$$

One obtains that

$$A(s)w = A_0 w + L_s^{-1}(-q^s(0) f'(p_s) w \dot{p}_s + \langle q^s, w \rangle X_0 f'(p_s)\dot{p}_s + X_0 f'(p_s) L_s w)$$

$$= A_0 w + L_s^{-1}(-q^s(0) f'(p_s) w \dot{p}_s + X_0 f'(p_s) w).$$

Set $\kappa(s) = -q^s(0) f'(p_s) w \dot{p}_s + X_0 f'(p_s) w$. It is easy to check that $\langle q^s, \kappa(s) \rangle = 0$, i.e., $\kappa(s) \in Q(s) \oplus [X_0]$. $A(s)$ is therefore well defined. Note that $\kappa(s) = L_s X_0 f'(p_s) w$. Using this relation, the following natural representation for $A(s)$ is obtained:

$$A(s)w = A_0 w + X_0 f'(p_s) w. \tag{3.4}$$

The structure of the original equation compared to the equation representing the flow on M_0 is essentially unchanged in the sense that in both instances the linear part consists of an unbounded differential operator and a bounded part. Eq. (3.4) is easily seen to be the one obtainable when applying the change of coordinates (3.3) to the equations obtained from the Floquet decomposition.

Remark 1. For each fixed $w \in M_0 \cap C^1$, the map $s \mapsto A(s)w$ is continuous. However, the operator $A(s): M_0 \cap C^1 \to BM$ is unbounded. To study periodic solutions of autonomous equations of the type $\dot{s} = 1 + f_1(s, w)$ and $\dot{w} = A(s)w + f_2(s, w)$, one could try to eliminate t by computing

$$\frac{dw}{ds} = A(s)w + W(s, w). \tag{3.5}$$

Both A and W are ω-periodic in s. However, the non-linear part, $W(s, w)$, is not bounded. For that reason Fredholm alternative-type arguments cannot be applied to show the existence of an ω-periodic solution of (3.5).

Remark 2. L_s^{-1} is bounded on $M_0(\varepsilon) = \{w \in M_0 : |w| < \varepsilon\}$.

To see that, let $\psi \in Q_1(s)$. Then $L_s^{-1}\psi = \phi \in M_0$ and $\phi = \alpha_s \dot{p}_s + \psi$. Thus, $|\phi| \leqslant |\alpha_s||\dot{p}_s| + |\psi|$. We only consider the portion of M_0 with $|\cdot|_{M_0} < \varepsilon$, therefore, $|\alpha_s| \leqslant k\varepsilon$,

$$|L_s^{-1}\psi| \leqslant \bar{k}\varepsilon + |\psi| \leqslant m_\varepsilon|\psi|.$$

The following lemma will play a major role in the proof of the invariant manifold theorem. Since the linear part of the equation on M_0 is represented by an unbounded operator, we need the following Lipschitz-type property in order to give estimates on the growth of the solution. Denote by $C^{k,1}(C, \mathbb{R}^n)$ the space of functions $f : C \to \mathbb{R}^n$ that are k-times differentiable and whose kth derivative is Lipschitz.

Lemma 2. *If we let $M_0^1 = M_0 \cap C^1$, and suppose $f \in C^{1,1}$, then*

$$||A(s_1) - A(s_2)||_{\mathfrak{L}(M_0^1, BM)} \leqslant C|s_1 - s_2|.$$

Proof. Let $w \in M_0^1$. Then $A(s_1)w - A(s_2)w = X_0[f'(p_{s_1}) - f'(p_{s_2})]w$. Therefore,

$$||A(s_1)w - A(s_2)w||_{BM} \leqslant l_{f'} l_p |s_1 - s_2||w|_C,$$

where $l_{f'}$ and l_p denote the Lipschitz constants of f' and p, respectively. □

3.3. Floquet multipliers

In this section, we relate the Floquet multipliers of the linear variational equation

$$\dot{y}_t = A(t)y_t, \qquad y_t \in C^1 \tag{3.6}$$

to those of

$$\dot{w} = A(t)w, \qquad w \in M_0, \tag{3.7}$$

where $A(t) = A_0 + X_0 f'(p_t)$. The solution operator of (3.7) is denoted by $T_A(t,s)$, while $T(t,s)$ denotes the solution operator of Eq. (3.6). First we need the following lemma.

Lemma 3. *Let $\bar{\phi} \in M_0$. Then the solution $T(t,s)L_s\bar{\phi}$ of* (3.6) *is given by $L_t T_A(t,s)\bar{\phi}$.*

Proof. It is easy to show that $L_t T_A(t,s)\bar{\phi}$ solves (3.6). We assumed that Γ is non-degenerate. This implies that $T(t,s)$ maps $Q_1(s)$ into $Q_1(t)$. From uniqueness it follows that $T(t,s)L_s\bar{\phi} = L_t T_A(t,s)\bar{\phi}$. □

Lemma 4. *Define $U_A(t) = T_A(t+\omega, t))$ and $U(t) = T(t+\omega, t)$. Then, for all $t \geqslant 0$, $U_A(t) = L_t^{-1} U(t) L_t$.*

Proof. From the definition of U_A we have $U_A(t)w(t) = w(t+\omega)$. Thus,

$$\begin{aligned} \alpha\dot{p}_{t+\omega} + L_{t+\omega}U_A(t)w(t) &= \alpha\dot{p}_{t+\omega} + L_{t+\omega}w(t+\omega) \\ &= U(t)(\alpha\dot{p}_t + L_t w(t)) \\ &= \alpha\dot{p}_t + U(t)L_t w(t). \end{aligned}$$

Since $\dot{p}_{t+\omega} = \dot{p}_t$, we obtain $L_{t+\omega}U_A(t)w(t) = L_t U_A(t)w(t) = U(t)L_t w(t)$. Therefore, $U_A(t) = L_t^{-1}U(t)L_t$. $\square$

Lemma 5. *If Γ is non-degenerate; that is, if* 1 *is a simple Floquet multiplier of* (3.6), *then all Floquet multipliers of* (3.7) *are different from* 1.

Proof. Consider $U_A = T_A(\omega, 0)$, $U_A w(t) = w(t+\omega)$. We want to show that 1 is not an eigenvalue of U_A. Suppose, on the contrary, that it is, i.e., there exists a $w(t) \neq 0 \in M_0$ such that $U_A w(t) = w(t)$. In other words, w is an ω-periodic solution of (3.7). Now consider $y_t = \alpha\dot{p}_t + L_t w(t)$. y_t solves (3.6). The ω-periodicity of $\dot{p}_t$, w and L_t imply that $y_t = y_{t+\omega}$. Since $w \neq 0$, there is no α such that $y_t = \alpha\dot{p}_t$. On the other hand, from the definition of L_t follows that $y_t \neq 0$. Thus, y_t is an eigenvector corresponding to the Floquet multiplier 1. As a consequence, 1 cannot be a simple Floquet multiplier of (3.6), which is a contradiction. $\square$

The set of Floquet multipliers, σ, of Eq. (3.6) can be split into subsets $\sigma = \sigma_- \cup \sigma_1 \cup \sigma_+$, where σ_- and σ_+ contain all Floquet multipliers inside and outside the unit circle, respectively. The set σ_1 is the set of all Floquet multipliers lying on the unit circle. Corresponding to the decomposition of σ, there exist projections $P_-(t)$, $P_+(t)$ and $P_1(t)$, such that any solution y of (3.6) can be decomposed into $y = y_+ + y_1 + y_-$, where $P_-(t)y = y_-$, $P_+(t)y = y_+$ and $P_1(t)y = y_1$, and there exist numbers $\beta > 0$, $M > 0$ and for every $v > 0$ a constant M_v such that

$$||T(t,s)P_-(s)||_{\mathfrak{L}(BC,BC)} \leqslant Me^{-\beta(t-s)} \quad \text{for } t \geqslant s, \tag{3.8}$$

$$||T(t,s)P_+(s)||_{\mathfrak{L}(BC,BC)} \leqslant Me^{\beta(t-s)} \quad \text{for } t \leqslant s, \tag{3.9}$$

$$||T(t,s)P_1(s)||_{\mathfrak{L}(BC,BC)} \leqslant M_v e^{v|t-s|} \quad \text{for } t \in \mathbb{R}. \tag{3.10}$$

It follows from their definition that the projections P_-, P_+ and P_1 are bounded.

4. An integral manifold theorem

In order to study invariant manifolds, one needs certain exponential estimates, or more precisely, exponential dichotomy, [4,11,12]. We give such estimates first for the equation $\dot{w} = A(t)w$ and then show that it is possible to obtain such estimates for the equation $\dot{w} = A(s)w$, where s satisfies a differential equation. Using these estimates, we prove the existence of an invariant manifold for general perturbations. One of the main conditions in the proof of the theorem is the presence of exponential dichotomy in the system.

Definition 5. Consider the linear system

$$\dot{z} = A(t)z, \tag{4.1}$$

where z is in some Banach space Z and $A \in C(\mathbb{R}, \mathfrak{L}(Z,Z))$. Denote by $Z(t,s)$ the solution operator for (4.1). System (4.1) is said to have an *exponential dichotomy* of type (α, β, K) on $\mathbb{R}$ if there exists a family of projections $P(t), Q(t) = I - P(t),\ t \in \mathbb{R}$, satisfying

(i) $P(t)Z(t,s) = Z(t,s)P(s)$ for $t, s \in \mathbb{R}$,
(ii) $||Z(t,s)P(s)||_{\mathfrak{L}(Z,Z)} \leqslant Ke^{-\alpha(t-s)}$ for $t \geqslant s$,
(iii) $||Z(t,s)Q(s)||_{\mathfrak{L}(Z,Z)} \leqslant Ke^{\beta(t-s)}$ for $s \geqslant t$,
where $\alpha > 0, \beta > 0, K \geqslant 1$.

Lemma 6. *Suppose that Γ is a hyperbolic periodic orbit and let M, β be as in* (3.8) *and* (3.9). *Then the equation*

$$\dot{w} = A(t)w \tag{4.2}$$

has an exponential dichotomy of type (β, β, K^2M) with the projections

$$\pi_+(t) = L_t^{-1}[I - P_1(t)]^{-1}P_+(t)[I - P_1(t)]L_t,$$

$$\pi_-(t) = L_t^{-1}[I - P_1(t)]^{-1}P_-(t)[I - P_1(t)]L_t,$$

where K is such that $\frac{1}{K}|z|_{M_0} \leqslant |[I - P_1(t)]L_t z|_{M_0} \leqslant K|z|_{M_0}$.

Proof. First we show that $\pi_-(t)T_A(t,s) = T_A(t,s)\pi_-(s)$ for all $t, s \in \mathbb{R}$. From $T(t,s)P_*(s) = P_*(t)T(t,s)$, $* = +, -, 1$, and after applying the previous lemma we obtain that for a given $\phi \in M_0$,

$$\begin{aligned}
&P_-(t)(I - P_1(t))L_t T_A(t,s)\phi \\
&\quad = P_-(t)(I - P_1(t))T(t,s)L_s\phi \\
&\quad = T(t,s)P_-(s)(I - P_1(s))L_s\phi \\
&\quad = T(t,s)(I - P_1(s))(I - P_1(s))^{-1}P_-(s)(I - P_1(s))L_s\phi \\
&\quad = (I - P_1(t))L_t T_A(t,s)L_s^{-1}(I - P_1(s))^{-1}P_-(s)(I - P_1(s))L_s\phi.
\end{aligned}$$

In the last step, we applied $T(t,s)L_s\psi = L_t T_A(t,s)\psi$ to $\psi = L_s^{-1}\ (I - P_1(s))^{-1}$ $P_-(s)(I - P_1(s))L_s\phi$.

To verify the exponential estimates, let $\phi \in M_0$. For all $t \geqslant s$

$$
\begin{aligned}
||T_A(t,s)\ \pi_-(s)\phi|| &= ||\pi_-(t)\ T_A(t,s)\phi|| \\
&= ||[(I - P_1(t))L_t]^{-1} P_-(t)(I - P_1(t))L_t T_A(t,s)\phi|| \\
&\leqslant K||P_-(t)(I - P_1(t))L_t T_A(t,s)\phi|| \\
&= K||P_-(t)(I - P_1(t))T_1(t,s)L_s\phi|| \\
&\leqslant KMe^{-\beta(t-s)}||(I - P_1(s))L_s\phi|| \\
&\leqslant K^2 Me^{-\beta(t-s)}||\phi||.
\end{aligned}
$$

The proof for π_+ is identical to the one given for π_-. $\square$

Remark 3. The exponential estimates remain valid also if we do not assume that Γ is hyperbolic. However, in that case it is no longer true that $\pi_+(t) = I - \pi_-(t)$.

Lemma 6 implies exponential dichotomy on $\mathbb{R}$ for system

$$
\dot{w} = A(s(t))w, \tag{4.3}
$$

where $\dot{s} = 1$ with $s(t_0) = \eta$ if Γ is hyperbolic. We intend to show that the same is true for the system (4.3) with $\dot{s} = 1 + g(t,s,w,\lambda)$ for some appropriate function g. In the following, Λ denotes a complete normed space. Assume that the following condition on g is satisfied:

(H1) $g : \mathbb{R} \times M_0 \times \mathbb{R} \times \Lambda \to M_0$ is continuous, bounded, $g(t,s,0,0) = 0$ and

$$
\begin{aligned}
&|g(t,s_1,w_1,\lambda_1) - g(t,s_2,w_2,\lambda_2)|_{M_0} \\
&\quad \leqslant N(|s_1 - s_2| + |w_1 - w_2|_{M_0} + |\lambda_1 - \lambda_2|_\Lambda).
\end{aligned}
$$

If g satisfies condition (H1), then if s_0 is the unique solution of $\dot{s_0} = 1$, and s is the unique solution of $\dot{s} = 1 + g(t,s(t),w,\lambda)$, $s(\tau) = s_0(\tau)$, the generalized Gronwall's inequality gives

$$
\begin{aligned}
|s(t) - s_0(t))| &\leqslant \int_\tau^t |g(\alpha, s(\alpha), w(\alpha), \lambda)|\, d\alpha \\
&\leqslant N \int_\tau^t (|w(\alpha)| + |\lambda|)\, d\alpha + N \int_\tau^t |s(\alpha) - s_0(\alpha)|\, d\alpha
\end{aligned}
$$

$$\begin{aligned}
&\leqslant N\int_\tau^t (|w(\alpha)|+|\lambda|)\,d\alpha + N^2\int_\tau^t (|w(\alpha)|+|\lambda|)e^{N(t-\alpha)}\,d\alpha\\
&\leqslant N\int_\tau^t e^{N|t-\alpha|}(|w(\alpha)|+|\lambda|)\,d\alpha\\
&\leqslant (\varepsilon+\lambda_0)(e^{N|t-\tau|}-1).
\end{aligned}$$

Consider the two systems

$$(*)\quad \begin{cases} \dot{s_0} = 1,\\ \dot{w} = A(s_0)w \end{cases} \quad \text{and} \quad (**)\quad \begin{cases} \dot{s} = 1+g(t,s,w,\lambda),\\ \dot{w} = A(s)w. \end{cases}$$

For $s_0(\tau)=s(\tau)$ we obtain that

$$\begin{aligned}
|A(s(t))-A(s_0(t))| &\leqslant c|s(t)-s_0(t))|\\
&\leqslant c(\varepsilon+\lambda_0)(e^{N|t-\tau|}-1).
\end{aligned} \tag{4.4}$$

Choose $h>0$ in such a way that $e^{Nh}(1-Nh)<1$. The previous calculation shows that for all $|t-\tau|<h$, we have

$$|A(s(t))-A(s_0(t))| < c(\varepsilon_0+\lambda_0)Nhe^{Nh} \tag{4.5}$$

provided that $s(\tau)=s_0(\tau)$. This estimate is sufficient to show that $(**)$ has exponential dichotomy for sufficiently small values of ε and λ_0. The proof involves the idea of discrete dichotomy and is as in [13,10].

4.1. Existence of an invariant manifold

There are many results in which the exponential dichotomy of a linear system is used to prove the existence of integral manifolds of a non-linear perturbation of that system [4,7,10,11,13]. We will present a proof as it applies to our situation and hope to limit the accumulation of constants resulting from a series of estimates to a minimum.

Definition 6. A surface S in the (t,z)-space is an integral manifold of a system of differential equations $\dot{z}=Z(z,t)$, if for any point P in S, the solution $z(t)$ through P is such that $(t,z(t))$ is in S for all time t in the domain of definition of the solution $z(t)$.

The system $\dot{s}=1,\ \dot{w}=A(s)w$, has an integral manifold S_0 in $\mathbb{R}\times\mathbb{R}\times M_0^1$ given by $S_0=\{(t,t,0): t\in\mathbb{R}\}$. We want to obtain a result on the existence of a manifold close to S_0 of the perturbed system

$$\begin{aligned}
&\dot{w} = A(s)w + f(t,s,w,\lambda),\\
&\dot{s} = 1+g(t,s,w,\lambda),\quad s(t_0)=\eta,
\end{aligned} \tag{4.6}$$

where g satisfies condition (H1), $f(t,s,0,0)=0$ and f satisfies the condition

(H2) $f:\mathbb{R}\times M_0\times\mathbb{R}\times\Lambda\to M_0$ is continuous, $f(t,s,0,0)=0$, and

$$|f(t,s_1,w_1,\lambda_1)-f(t,s_2,w_2,\lambda_2)|<M(|w_1+w_2|+|\lambda_1+\lambda_2|)$$

$$|w_1-w_2|+M|\lambda_1+\lambda_2|||s_1-s_2|+M|\lambda_1-\lambda_2|$$

Theorem 1. *Suppose Eq.* (4.3) *has exponential dichotomy of type* (α,β,K) *on* $\mathbb{R}$ *and is of bounded growth and decay with* (C,μ) *uniformly in* (t_0,η). *If* α, β *are sufficiently large and if* N *in condition* (H1) *is such that* $N<\min(\alpha,\beta)$, *then there is a* λ_0 *such that for* $|\lambda|\leqslant\lambda_0$ *system* (4.6) *has an integral manifold near* $\mathbb{R}\times\mathbb{R}\times\{0\}$ *such that*

$$S_\lambda=\{\,(t,\eta,\sigma(t,\eta,\lambda))\in\mathbb{R}\times\mathbb{R}\times M_0;\ t\in\mathbb{R},\ \eta\in\mathbb{R}\}$$

is an integral manifold for system (4.6) *and*

(i) σ *is Lipschitz continuous in* (η,λ) *uniformly with respect to* $t\in\mathbb{R}$
(ii) $\sup\{|\sigma(t,\eta,\lambda)|;\ t\in\mathbb{R},\ \eta\in\mathbb{R}\}=O(|\lambda|)$.

Proof. Without loss of generality, we may assume that system (4.3) with $\dot{s}=1+g(t,s,w,\lambda)$ has exponential dichotomy of type (α,β,K). Denote by $\tilde{P}$ the projection corresponding to the exponential dichotomy. Define an operator $\mathfrak{T}$ by

$$\begin{aligned}(\mathfrak{T}w)(t)=&\int_{-\infty}^{t}T_A(t,\tau)\tilde{P}(\tau)f(\tau,s(\tau),w(\tau),\lambda)\,d\tau\\&-\int_{t}^{\infty}T_A(t,\tau)(I-\tilde{P}(\tau))f(\tau,s(\tau),w(\tau),\lambda)\,d\tau.\end{aligned}$$

Note that every bounded solution of (4.3) is of that form. We claim that $\mathfrak{T}$ is a contraction in the supremum norm in the space $M_0^\varepsilon=\{w:\mathbb{R}\to M_0\mid ||w||<\varepsilon\}$, where $||w||=\sup_{t\in\mathbb{R}}|w(t)|_C$.

First we will show that $\mathfrak{T}:M_0^\varepsilon\to M_0^\varepsilon$. Let $w\in M_0^\varepsilon$. Then,

$$\begin{aligned}&\int_{-\infty}^{t}||T_A(t,\tau)\tilde{P}(\tau)||\,|f(\tau,s(\tau),w(\tau),\lambda)|\,d\tau\\&\leqslant KM[(||w||+|\lambda|)||w||+|\lambda|]\int_{-\infty}^{t}e^{-\alpha(t-\tau)}\,d\tau\\&=KM(\varepsilon^2+\lambda_0\varepsilon+\lambda_0)\frac{1}{\alpha},\end{aligned}$$

$$\begin{aligned}
&\int_t^\infty ||T_A(t,\tau)(I-\tilde{P}(\tau))||\,|f(\tau,s(\tau),w(\tau),\lambda)|\,d\tau \\
&\quad \leqslant KM[(||w||+|\lambda|)||w||+|\lambda|]\int_t^\infty e^{\beta(t-\tau)}\,d\tau \\
&\quad = KM(\varepsilon^2+\lambda_0\varepsilon+\lambda_0)\frac{1}{\beta}.
\end{aligned}$$

If $KM(\varepsilon^2+\lambda_0\varepsilon+\lambda_0)(\frac{1}{\alpha}+\frac{1}{\beta})<\varepsilon$, then $\mathfrak{T}$ maps from M_0^ε into M_0^ε.

To show that $\mathfrak{T}$ is a contraction, assume that $w_1, w_2 \in M_0^\varepsilon$. For $t\in\mathbb{R}$, using the same estimates as before,

$$\begin{aligned}
&|(\mathfrak{T}w_1)(t)-(\mathfrak{T}w_2)(t)| \\
&\quad \leqslant \int_{-\infty}^t ||T_A(t,\tau)\tilde{P}(\tau)||\,|f(\tau,s(\tau),w_1(\tau),\lambda)-f(\tau,s(\tau),w_2(\tau),\lambda)|\,d\tau \\
&\qquad + \int_t^\infty ||T_A(t,\tau)(I-\tilde{P}(\tau))||\,|f(\tau,s(\tau),w_1(\tau),\lambda)-f(\tau,s(\tau),w_2(\tau),\lambda)|\,d\tau \\
&\quad \leqslant KM(|w_1+w_2|+|\lambda_1+\lambda_2|)||w_1-w_2||\left(\frac{1}{\alpha}+\frac{1}{\beta}\right).
\end{aligned}$$

Therefore, if $2KM(\varepsilon+\lambda_0)(\frac{1}{\alpha}+\frac{1}{\beta})<1$, we obtain that $\mathfrak{T}$ is a contraction. This implies the existence of a unique fixed point of $\mathfrak{T}$ in M_0^ε which, on the other hand, proves the existence of a unique solution $\tilde{w}(\cdot\,;s,\lambda)$ of Eq. (4.3) with norm $||\tilde{w}||<\varepsilon$.

Denote by $\tilde{s}(t;t_0,\eta,w,\lambda)$ the solution of $\dot{s}=1+g(t,s,w(t),\lambda)$ with initial condition $s(t_0)=\eta$. Condition (H1) implies the following estimate for two different solutions $\tilde{s}_1$ and $\tilde{s}_2$, where $\tilde{s}_i(t)=\tilde{s}(t;t_0,\eta_i,w_i(t),\lambda_i)$.

$$\begin{aligned}
|\tilde{s}_1(t)-\tilde{s}_2(t)| &\leqslant |\eta_1-\eta_2| + \int_{t_0}^t |g(\tau,s_1(\tau),w_1(\tau),\lambda_1) \\
&\qquad - g(\tau,s_2(\tau),w_2(\tau),\lambda_2)|\,d\tau \\
&\overset{(H1)}{\leqslant} |\eta_1-\eta_2| + N\int_{t_0}^t (|w_1(\tau)-w_2(\tau)|+|\lambda_1-\lambda_2|)\,d\tau \\
&\qquad + N\int_{t_0}^t |s_1(\tau)-s_2(\tau)|\,d\tau \\
&\overset{(*)}{\leqslant} |\eta_1-\eta_2|e^{N|t-t_0|} + N \\
&\qquad \times \int_{t_0}^t (|w_1(\tau)-w_2(\tau)|+|\lambda_1-\lambda_2|)e^{N|t-\tau|}\,d\tau \\
&\leqslant |\eta_1-\eta_2|e^{N|t-t_0|} + (||w_1-w_2||+|\lambda_1-\lambda_2|)(e^{N|t-t_0|}-1).
\end{aligned}$$

Inequality $(*)$ is an application of Gronwall's lemma and integration by parts.

The next step is to show the existence of a mutual solution of system (4.6). We will again prove this by showing that there is a contraction whose unique fixed point forms that solution.

Define $\mathfrak{S}: M_0^\varepsilon \times \mathbb{R} \times \mathbb{R} \times \Lambda \to M_0^\varepsilon$ by

$$\mathfrak{S}(w, t_0, \eta, \lambda)(t) = \tilde{w}(t, \tilde{s}(t; t_0, \eta, w, \lambda), \lambda).$$

For any fixed $t_0 \in \mathbb{R}$ and any $t \in \mathbb{R}$

$$\begin{aligned}
&|\mathfrak{S}(w_1, t_0, \eta_1, \lambda_1)(t) - \mathfrak{S}(w_2, t_0, \eta_2, \lambda_2)(t)| \\
&\overset{(H2)}{\leqslant} KM[((2\varepsilon + 2\lambda_0)||w_1 - w_2|| + |\lambda_1 - \lambda_2|) \int_{-\infty}^{t} e^{-\alpha(t-\tau)}\, d\tau \\
&\quad + 2\lambda_0 \int_{-\infty}^{t} e^{-\alpha(t-\tau)} |\tilde{s}_1(\tau) - \tilde{s}_2(\tau)|\, d\tau] \\
&\quad + KM\left[\text{same for } \int_t^\infty e^{\beta(t-\tau)}\, d\tau\right] \\
&\leqslant 2KM\lambda_0(I_1 + I_2)|\eta_1 - \eta_2| \\
&\quad + 2KM\left[\varepsilon\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + \lambda_0(I_1 + I_2)\right]||w_1 - w_2|| \\
&\quad + KM\left[(1 - 2\lambda_0)\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + 2\lambda_0(I_1 + I_2)\right]|\lambda_1 - \lambda_2|,
\end{aligned}$$

where $I_1 = \int_{-\infty}^{t} e^{-\alpha(t-\tau)+N|\tau - t_0|}\, d\tau$ and $I_2 = \int_t^\infty e^{\beta(t-\tau)+N|\tau-t_0|}\, d\tau$.

For $t \geqslant t_0$ and under the condition that $0 < N < \min(\alpha, \beta)$

$$I_1 = \frac{1}{\alpha - N} e^{-\alpha|t-t_0|} + \frac{1}{\alpha + N}[e^{N|t-t_0|} - e^{-\alpha|t-t_0|}] \quad \text{and} \quad I_2 = \frac{1}{\beta - N} e^{N|t-t_0|}.$$

For $t \leqslant t_0$ one obtains

$$I_1 = \frac{1}{\alpha - N} e^{N|t-t_0|} \quad \text{and} \quad I_2 = \frac{1}{\beta - N} e^{-\beta|t-t_0|} + \frac{1}{\beta + N}[e^{N|t-t_0|} - e^{-\beta|t-t_0|}].$$

For $x \in M_0$ define the weighted norm $|||x|||_N = \sup_{t \in \mathbb{R}} e^{-N(t-t_0)}||x||$. Setting $\eta_1 = \eta_2$ and $\lambda_1 = \lambda_2$, we obtain that $\mathfrak{S}$ is a contraction in the $|||\cdot|||_N$-norm provided that

$$\frac{1}{\alpha + N} + \frac{1}{\alpha - N} < \frac{1}{KM(\varepsilon + \lambda_0)} \quad \text{and} \quad \frac{1}{\beta + N} + \frac{1}{\beta - N} < \frac{1}{KM(\varepsilon + \lambda_0)}.$$

The above estimates also show that $\mathfrak{S}$ is Lipschitz in the $|||\cdot|||$-norm in η and λ uniformly in t.

Let $w^*(\cdot\,; t_0, \eta, \lambda)$ be the unique fixed point of $\mathfrak{S}$. Clearly, w^* solves $\dot{w} = A(s)w$. Let

$$s^*(t; t_0, \eta) = \tilde{s}(t; t_0, \eta, w^*(t; t_0, \eta.\lambda), \lambda).$$

(s^*, w^*) forms a solution of (4.6) with $s^*(t_0) = \eta$ and $||w^*|| < \varepsilon$. Define $\sigma(t_0, \eta, \lambda) = w^*(t_0; t_0, \eta, \lambda)$ for $t_0 \in \mathbb{R}$, $\eta \in \mathbb{R}$ and $\lambda \in \Lambda$. If for some pair (s, w), Eqs. (4.6) are satisfied, then uniqueness implies that

$$s(t) = s^*(t; t_0, s(t_0), \lambda), \quad w(t) = w^*(t; t_0, s(t_0), \lambda) \quad \text{for all } t, t_0.$$

Thus, we have $w(t_0) = \sigma(t_0, s(t_0), \lambda)$. On the other hand, if $w(t_0) = \sigma(t_0, s(t_0), \lambda)$ for some t_0, then the solution with this initial value exists for all t. Therefore, $w(t) = \sigma(t, s(t), \lambda)$. Thus,

$$S_\lambda = \{(t, s, w) \mid w(t) = \sigma(t, s(t), \lambda)\}$$

is an integral manifold that is Lipschitz in λ and η. Part (ii) of the statement follows from the estimate $|\mathfrak{T}w(t)| \leqslant KM[(||w|| + |\lambda|)||w|| + |\lambda|](\frac{1}{\alpha} + \frac{1}{\beta})$. $\square$

Remark 4. If $s \mapsto A(s), f(\cdot, s, \cdot, \cdot), g(\cdot, s, \cdot, \cdot)$ are ω-periodic, then $\sigma(t, s, \lambda)$ is ω-periodic in s. This is clear after noting that $s(t; t_0, \eta + \omega) = s(t; t_0, \eta) + \omega$ which follows from uniqueness.

Remark 5. If the function f is periodic in t (almost periodic in t), then $\sigma(t, s, \lambda)$ is periodic in t with the same period (almost periodic with the same module as f).

Remark 6. If f and g are independent of t, i.e., the equations are autonomous, then σ too is independent of t. Let $\omega(\lambda)$ be the solution of $s(\omega(\lambda)) = \omega$ where s is the solution of equation $\dot{s} = 1 + g(s, w, \lambda)$ with initial value $s(0) = 0$. The function $\omega : \Lambda \to \mathbb{R}$ is continuous in λ, and $\omega(0) = \omega$. Thus, if the perturbation of $\dot{x}(t) = f(x_t)$ is autonomous, then the integral manifold S_λ is of the form $S_\lambda = \{(t, s, w) || \; w(t) = \sigma(s(t), \lambda)\}$ and w is $\omega(\lambda)$-periodic in t.

5. The new coordinates work well

In this section, we consider perturbations of the equation

$$\frac{d}{dt} Dx_t = f(x_t), \tag{5.1}$$

where D is a stable operator with a representation as in Section 2.2. The right-hand side f is supposed to be twice differentiable with its second derivative being Lipschitz, $f \in C^{2,1}(C, \mathbb{R}^n)$. Furthermore, Eq. (5.1) is assumed to have a non-trivial periodic orbit $\Gamma = \{p_t : 0 \leqslant t \leqslant \omega\}$ of minimal period ω which is such that condition (B) is satisfied. Consider a perturbation of Eq. (5.1) of the form

$$\frac{d}{dt} Dx_t = f(x_t) + g(t, x_t, \lambda), \tag{5.2}$$

where $\lambda \in \Lambda$, Λ is a bounded subset of $\mathbb{R}^d$, and $g(t, \phi, 0) = 0$, $\phi \in C$. In abstract form, Eq. (5.2) becomes

$$\dot{x}_t = A_0 x_t + X_0[f(x_t) + g(t, x_t, \lambda)].$$

The extended infinitesimal generator is given by $A_0\phi = \phi' - X_0 D\phi'$, for $\phi \in C^1$. In the spirit of our previous discussion, introduce coordinates (s, w) in a neighborhood of Γ, $x_t = p_s + L_s w$. This transformation is well defined for $|w|_C < \varepsilon$, with ε sufficiently small. Eq. (5.2) becomes

$$\dot{s} = 1 + h_1(t, s, w, \lambda), \tag{5.3}$$

$$\dot{w} = A(s)w + h_2(t, s, w, \lambda), \tag{5.4}$$

where $A(s) = A_0 + X_0 f'(p_s)$ and

$$\begin{aligned} h_1(t, s, w, \lambda) = {} & [1 + \langle q^s, \partial_s L_s w \rangle_s]^{-1}(-\langle q^s, \partial_s L_s w \rangle_s \\ & + q^s(0)[f(p_s + L_s w) - f(p_s) + g(t, p_s + L_s w, \lambda)]), \end{aligned}$$

$$\begin{aligned} h_2(t, w, s, \lambda) = {} & L_s^{-1}(-\partial_s L_s w\, h_1(t, s, w, \lambda) \\ & + X_0[f(p_s + L_s w) - f(p_s) - f'(p_s) L_s w + g(t, p_s + L_s w, \lambda)]). \end{aligned}$$

The inverse $[1 + \langle q^s, \partial_s L_s w \rangle_s]^{-1}$ exists if $|\langle q^s, \partial_s L_s w \rangle_s|$ is strictly less than one. This can be ensured provided that $\partial_s L_s w$ is bounded and small. One can verify by direct computation that there exists a $K = K(f)$ such that $|\partial_s L_s w| \leqslant K|w|_C$, provided that $f \in C^2(C, \mathbb{R}^n)$.

Clearly, $h_1(t, s, 0, 0) = 0$ and $h_2(t, s, 0, 0) = 0$. Even though we are only interested in a neighborhood of $w = 0$, $\lambda = 0$, we follow the usual procedure and introduce new functions $\tilde{h}_1$ and $\tilde{h}_2$, which will be identical to h_1 and h_2 in a neighborhood of $w = 0$ and $\lambda = 0$, and satisfy conditions (H1) and (H2) globally. More precisely, let $\chi_w : M_0 \to [0, 1]$, $\chi_\lambda : \Lambda \to [0, 1]$ be C^∞ functions satisfying

$$\chi_*(x) = \begin{cases} 1, & |x| \leqslant 1, \\ 0, & |x| \geqslant 2 \end{cases}$$

for $* = w, \lambda$. For $\varepsilon > 0$ and $\lambda_0 > 0$ define

$$\begin{aligned} \tilde{h}_1(t, w, s, \lambda) &= h_1\left(t, w \cdot \chi_w\left(\frac{w}{\varepsilon}\right), s, \lambda \cdot \chi_\lambda\left(\frac{\lambda}{\lambda_0}\right)\right), \\ \tilde{h}_2(t, w, s, \lambda) &= h_2\left(t, w \cdot \chi_w\left(\frac{w}{\varepsilon}\right), s, \lambda \cdot \chi_\lambda\left(\frac{\lambda}{\lambda_0}\right)\right). \end{aligned}$$

If ε and λ_0 are chosen sufficiently small, then the functions $\tilde{h}_1$ and $\tilde{h}_2$ satisfy conditions (H1) and (H2) and we are able to apply Theorem 1 to obtain the following theorem.

5.1. The hyperbolic case

Theorem 2. *Suppose $f \in C^{2,1}(C, \mathbb{R}^n)$ and suppose that g is Lipschitz uniformly in t, $g(t, \phi, 0) = 0$, $\phi \in C$. Assume that Eq.* (5.1) *has a non-trivial hyperbolic periodic orbit Γ of minimal period ω. Then there is a $\lambda_0 \in \Lambda$ such that for all $|\lambda| < \lambda_0$, Eq.* (5.2) *has an integral manifold S_λ near $\mathbb{R} \times \Gamma$ given by*

$$S_\lambda = \{(t, x_t) \mid t \in \mathbb{R}, x_t = p_s + L_s\sigma(t, s, \lambda)\},$$

where σ is as in Theorem 1. *Furthermore, σ is ω-periodic in s. The function σ is periodic in t* (*almost periodic in t*) *if g is periodic in t with the same period as g* (*almost periodic in t with the same module as g*).

Proof. The proof that h_1 satisfies condition (H1) and that h_2 satisfies condition (H2) for $|\lambda| < \lambda_0$ and $|w| < \varepsilon$ consists of a series of standard but tedious estimates and is omitted. Since Γ is hyperbolic, Lemma 6 implies exponential dichotomy of the equation $\dot{w} = A(s)w$ with $\dot{s} = 1$ and consequently that of $\dot{w} = A(s)w$ with $\dot{s} = 1 + h_1$. The existence of the integral manifold and its properties follow from Theorem 1 and the remarks thereafter. $\square$

5.2. The non-degenerate case

Suppose the periodic orbit Γ of Eq. (5.1) is non-degenerate or such that condition (B) is satisfied. This means that there is a possibility of having more Floquet multipliers on the unit circle. Since the operator D is assumed to be stable, there can be at most finitely many Floquet multipliers on the unit circle. In Section 1.2, we saw that it is possible to decompose the phase space C according to a set Λ_1 of Floquet multipliers. Choose $\Lambda_1 = \{1, \mu_1, \mu_2, \ldots\}$ and let $\Lambda = \{\mu_1, \mu_2, \ldots\}$. Following the notation from Section 2.5, Λ_1 induces a decomposition of the space C into sets $C = [\dot{p}_s] \oplus E_\Lambda(s) \oplus Q_{\Lambda_1}(s)$ for $0 \leqslant s \leqslant \omega$. We want to decompose M_0 in a similar way. Define

$$M_0^\Lambda(t) = L_{s(t)}^{-1} E_\Lambda(s(t)),$$

$$M_0^Q(t) = L_{s(t)}^{-1} Q_{\Lambda_1}(s(t)).$$

Since $E_\Lambda(s) \oplus Q_{\Lambda_1}(s) = Q_1(s)$ and L_s is an isomorphism on its range, we obtain a decomposition of $M_0 = M_0^\Lambda(t) \oplus M_0^Q(t)$ for all t such that $0 \leqslant s(t) \leqslant \omega$. The projection

$$\pi_{M_0^\Lambda}(t) : M_0 \to M_0^\Lambda(t)$$

is given by

$$\pi_{M_0^\Lambda}(t) = L_{s(t)}^{-1}\pi_{E_\Lambda}(s(t))L_{s(t)}.$$

It is known that $\pi_{E_\Lambda}(s)\phi = \Phi_\Lambda(s)\langle \Psi_\Lambda(s), \phi\rangle_s$ for $\phi \in C$, and $\langle \Psi_\Lambda(s), X_0\rangle_s = \Psi_\Lambda(s)(0)$. Define $\Omega_\Lambda(0)$ by $L_{s(0)}\Omega_\Lambda(0) = \Phi_\Lambda(s(0))$. Then,

$$\begin{aligned} L_{s(t)}T_\Lambda(t,0)\Omega_\Lambda(0) &= T(s(t),s(0))L_{s(0)}\Omega_\Lambda(0) \\ &= T(s(t),s(0))\Phi_\Lambda(s(0)) \\ &= \Phi_\Lambda(s(t)). \end{aligned}$$

Claim. $\Omega_\Lambda(t) =^{\text{def}} T_\Lambda(t,0)\Omega_\Lambda(0)$ *forms a basis of* M_0^Λ.

Proof. Let $w_\Lambda \in M_0^\Lambda$. Then there exists a $q(s(t)) \in E_\Lambda(s(t))$ such that $w_\Lambda = L_{s(t)}^{-1}q(s(t))$. Since $\Phi_\Lambda(s(t))$ is a basis for $E_\Lambda(s(t))$, we can write $q(s(t)) = \Phi_\Lambda(s(t))a$ for some d_Λ-vector a. All together, $w_\Lambda = L_{s(t)}^{-1}\Phi_\Lambda(s(t))a = \Omega_\Lambda(t)a$. $\square$

Define $P(s) = \Phi_\Lambda(s)e^{-B_\Lambda s}$ and note that

$$\Omega_\Lambda(t)e^{-B_\Lambda s(t)} = L_{s(t)}^{-1}\Phi_\Lambda(s(t))e^{-B_\Lambda s(t)} = L_{s(t)}^{-1}P(s(t)).$$

Consider the perturbed equation (5.2) together with the change of variables $x_t = p_s + L_s w,\ w \in M_0$. Any $w \in M_0$ can be written as $w = w_\Lambda + \bar{w}$, where $w_\Lambda \in M_0^\Lambda$ and $\bar{w} \in M_0^Q$. The functions w_Λ and $\bar{w}$ satisfy the equations

$$\begin{aligned} \dot{w}_\Lambda &= A(s)w_\mu + L_s^{-1}\pi_{E_\Lambda}(s)L_s h_2(t,s,w_\Lambda+\bar{w},\lambda), \\ \dot{\bar{w}} &= A(s)\bar{w} + (I - L_s^{-1}\pi_{E_\Lambda}(s)L_s)\ h_2(t,s,w_\Lambda+\bar{w},\lambda). \end{aligned} \tag{5.5}$$

Let y be a vector in $\mathbb{R}^{d_\Lambda}$ such that $w_\Lambda = \Omega_\Lambda(t)e^{-B_\Lambda s(t)}y$. Then, after setting $B = B_\Lambda$, we obtain the following equations for $s,\ \bar{w}$, and y.

$$\dot{s} = 1 + S(t,s,\bar{w},y,\lambda), \tag{5.6}$$

$$\dot{y} = By + Y(t,s,\bar{w},y,\lambda), \tag{5.7}$$

$$\dot{\bar{w}} = A(s)\bar{w} + W(t,s,\bar{w},y,\lambda), \tag{5.8}$$

where

$$\begin{aligned} S(t,s,\bar{w},y,\lambda) &= 1 + h_1(t,s,\bar{w} + L_s^{-1}P(s)y,\lambda), \\ W(t,s,\bar{w},y,\lambda) &= [I - L_s^{-1}\pi_{E_\Lambda}(s)L_s]h_2(t,s,\bar{w} + L_s^{-1}P(s)y,\lambda), \\ Y(t,s,\bar{w},y,\lambda) &= L_s^{-1}\pi_{E_\Lambda}(s)L_s h_2(t,s,\bar{w} + L_s^{-1}P(s)y,\lambda)e^{Bs}. \end{aligned}$$

All eigenvalues of the matrix B have zero real part. One can show that, for any $v > 0$, there exists a K_v such that $|e^{Bt}| < K_v e^{v|t|}$. Without loss of generality, we may assume that $K_v > 1$, thus, $|B| < v \ln K_v$. The functions S and Y satisfy condition (H1) with constants N_S and N_Y, respectively. This guarantees the existence of a unique solution of Eqs. (5.6) and (5.7). Denote by $s(t; t_0, \eta, y, w, \lambda)$ the solution of Eq. (5.6) with initial condition $s(t_0) = \eta$, and by $y(t; t_0, y^0, s, w, \lambda)$ the solution of Eq. (5.7) with initial condition $y(t_0) = y^0$.

Eq. (5.8) represents the hyperbolic part, i.e., the part corresponding to those Floquet multipliers which do not lie on the unit circle. Thus, the equation $\dot{\bar{w}} = A(t)\bar{w}$ has exponential dichotomy. Consequently, also equation $\dot{\bar{w}} = A(s)\bar{w}$ with $\dot{s} = 1 + S(t, s, \bar{w}, y, \lambda)$ and $\dot{y} = 0$ has exponential dichotomy.

Let r be a vector in $\mathbb{R}^{d_A+1}$. Then, for $r = (s, y), |r| = |s| + |y|$, $r_i(t) = (s(t; t_0, \eta_i, y_i, w_i, \lambda_i), y(t; t_0, y_i^0, s_i, w_i, \lambda_i))$, and $r_i^0 = (\eta_i, y_i^0)$ the following estimate holds:

$$|r_1(t) - r_2(t)| \leqslant |r_1^0 - r_2^0| e^{(N_S + N_Y + |B|)|t-t_0|} + (N_S + N_Y + |B|) \times \int_{t_0}^{t} (|w_1(\tau) - w_2(\tau)| + |\lambda_1 - \lambda_2|) e^{(N_S + N_Y + |B|)|t-\tau|} \, d\tau.$$

This estimate is sufficient to show that $\dot{\bar{w}} = A(s)\bar{w}$ with Eqs. (5.6) and (5.7) has exponential dichotomy. Suppose the dichotomy is of type (α, β, M). The proof of Theorem 1 remains valid if we replace the condition $N < \min(\alpha, \beta)$ by $(N_S + N_Y + |B|) < (N_S + N_Y + v \ln K_v) < \min(\alpha, \beta)$.

As a consequence, there exists an invariant manifold near $\mathbb{R} \times \mathbb{R} \times \{0\} \times \{0\}$ given by

$$S_\lambda = \{(t, s, \bar{w}, y) : \bar{w} = \sigma(t, s, y, \lambda),\ t \in \mathbb{R},\ s \in \mathbb{R},\ y \in \mathbb{R}^{d_A}\}$$

for $|\lambda| < \lambda_0$. As before, σ is ω-periodic in s and Lipschitz in (s, y, λ) uniformly in t.

The above system of Eqs. (5.6)–(5.8) in a neighborhood of $(y, \bar{w}) = (0, 0)$ and for $|\lambda| < \lambda_0$ can therefore be reduced to the system

$$\dot{s} = 1 + \tilde{S}(t, s, y, \lambda), \tag{5.9}$$

$$\dot{y} = By + \tilde{Y}(t, s, y, \lambda), \tag{5.10}$$

near $y = 0$ and for $|\lambda| < \lambda_0$, where $\tilde{S}(t, s, y, \lambda) = S(t, s, \sigma(t, s, y, \lambda), y, \lambda)$ and $\tilde{Y}(t, s, y, \lambda) = Y(t, s, \sigma(t, s, y, \lambda), y, \lambda)$. Consider a solution $s(t, \lambda) = s(t; t_0, \eta, y^0, \lambda)$, $y(t, \lambda) = y(t; t_0, \eta, y^0, \lambda)$ of the above system. Then, for $|\lambda| < \lambda_0$, the solution x of Eq. (5.2) near Γ is given by

$$x_t = p_{s(t,\lambda)} + L_{s(t,\lambda)}(\sigma(t, s(t, \lambda), y(t, \lambda), \lambda) + L_{s(t,\lambda)}^{-1} P(s(t, \lambda)) y(t, \lambda)).$$

This representation is ω-periodic in s. We have the following result.

Theorem 3. *Suppose $f \in C^{2,1}(C, \mathbb{R}^n)$ and suppose that g is Lipschitz uniformly in t, $g(t, \phi, 0) = 0$, $\phi \in C$. Assume that Eq.* (5.1) *has a non-trivial periodic orbit Γ of minimal period ω which is such that condition* (B) *is satisfied. Then, for $|\lambda < \lambda_0|$, Eq.* (5.2) *has an integral manifold S_λ near $\mathbb{R} \times \Gamma$ given by*

$$S_\lambda = \{(t, x_t) \mid t \in \mathbb{R}, x_t = p_{s(t,\lambda)} + L_{s(t,\lambda)}(\sigma(t, s(t, \lambda), y(t, \lambda), \lambda) + L^{-1}_{s(t,\lambda)} P(s(t, \lambda))y(t, \lambda))\},$$

where σ is ω-periodic in s. The function σ is periodic in t (*almost periodic in t*) *if g is periodic in t with the same period as g* (*almost periodic in t with the same module as g*).

Remark 7. This theorem also recovers a theorem for autonomous perturbations of retarded FDE with a non-degenerate orbit given in [9, p. 325]. There, the function $F(\phi, \lambda) = f(\phi) + g(\phi, \lambda)$ was assumed to be in C^1. The proof involved a change of the phase space and made use of the Fredholm alternative. For that reason, it was not possible to give a representation of the integral manifold. It was also not possible to find the period explicitly.

5.3. Autonomous equations

In this section, we will discuss autonomous perturbations of Eq. (5.1), i.e., equations of the form

$$\frac{d}{dt} Dx_t = f(x_t) + g(x_t, \lambda), \tag{5.11}$$

where the function g is Lipschitz and $g(\phi, 0) = 0$, $\phi \in C$. Under these assumptions, the integral manifold S_λ is described by $\bar{w} = \sigma(s, y, \lambda)$, that is, σ is independent of t as well, and σ is ω-periodic in s. Consequently, the functions $\tilde{S}$ and $\tilde{Y}$ in (5.9) and (5.10) are also independent of t, $\tilde{S}(t, s, y, \lambda) = \tilde{S}(s, y, \lambda)$ and $\tilde{Y}(t, s, y, \lambda) = \tilde{Y}(s, y, \lambda)$. Furthermore, $\tilde{S}$ and $\tilde{Y}$ are ω-periodic in s, $\tilde{S}(s, 0, 0) = 0$ and $\tilde{Y}(s, 0, 0) = 0$.

For small values of λ and y, $\dot{s} > 0$, and therefore the equation

$$\frac{dy}{ds} = By + \Upsilon(s, y, \lambda) \tag{5.12}$$

is well defined and $\Upsilon(s + \omega, y, \lambda) = \Upsilon(s, y, \lambda)$, $\Upsilon(s, 0, 0) = 0$. The problem of finding periodic solutions of Eq. (5.11) can now be reduced to studying ω-periodic solutions of Eq. (5.12). If $y(s)$ is an ω-periodic solution of (5.12), then the original equation (5.11) has a periodic solution given by

$$x_t = p_s + L_s(\sigma(s, y(s), \lambda) + L_s^{-1} P(s)y(s)).$$

The period of x_t is the value $\omega(\lambda)$ for which $s(\omega(\lambda)) = \omega$, where s is the unique solution of the equation $\dot{s} = 1 + \tilde{S}(s, y(s), \lambda)$ with initial condition $s(0) = 0$. The function $\omega : \Lambda \to \mathbb{R}$ is continuous and $\omega(0) = \omega$. That is, if Eq. (5.12) has an ω-periodic solution, then the integral manifold S_λ in Theorem 3 is then given by $S_\lambda = \{(t, x_t) \mid t \in \mathbb{R}, x_t = p_s + L_s(\sigma(s, y(s, \lambda), \lambda) + L_s^{-1}P(s)y(s, \lambda))\}$; that is, S_λ is described by a cylinder $\mathbb{R} \times \Gamma_\lambda$, where Γ_λ is a periodic orbit of period $\omega(\lambda)$ of Eq. (5.2).

The existence of an ω-periodic solution $y(s)$ (5.12) depends largely on the matrix B. If B satisfies that $I - e^{B\omega}$ is invertible, then the equation $\frac{dy}{ds} = By$ has no ω-periodic solution. One can then apply Fredholm alternative-type arguments to show the existence of a solution that is ω-periodic.

If the periodic orbit Γ is hyperbolic, the integral manifold given in Theorem 2 behaves like a saddle. In the case when Γ is non-hyperbolic and the corresponding matrix B satisfies condition (B), the flow on the center manifold of Γ is described by Eq. (5.12). One can use classical methods as known for ordinary differential equations to characterize the dynamics on the center manifold, see [1] and the references therein.

References

[1] S.-N. Chow, J.K. Hale, in: Methods of Bifurcation Theory, Grundlehren der Mathematischen Wissenschaften, Vol. 251, Springer, New York, 1982.

[2] S.-N. Chow, J. Mallet-Paret, Integral averaging and bifurcation, J. Differential Equations 26 (1977) 112–159.

[3] A.M. Cruz, Jack K. Hale, Stability of functional differential equations of neutral type, J. Differential Equations 7 (1970) 334–355.

[4] W.A. Coppel, in: Dichotomies in Stability Theory, Lecture Notes in Mathematics, Vol. 629, Springer, New York/Berlin, 1978.

[5] T. Faria, L. Magalhães, Normal forms for retarded functional-differential equations and applications to Bogdanov–Takens singularity, J. Differential Equations 122 (1995) 201–224.

[6] T. Faria, L. Magalhães, Normal forms for retarded functional-differential equations with parameters and applications to Hopf bifurcation, J. Differential Equations 122 (1995) 181–200.

[7] J.K. Hale, Ordinary Differential Equations, 2nd Edition, Robert E. Krieger Publishing Co., Huntington, NY, 1980.

[8] J.K. Hale, J. Scheurle, Smoothness of bounded solutions of nonlinear evolution equations, J. Differential Equations 56 (1985) 142–163.

[9] J.K. Hale, S.M. Verduyn Lunel, in: Introduction to Functional-Differential Equations, Applied Mathematical Sciences, Vol. 99, Springer, New York, 1993.

[10] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Springer, Berlin, 1981.

[11] K.J. Palmer, A perturbation theorem for exponential dichotomies, Proc. Roy. Soc. Edinburgh Sect. A 103 (1987).

[12] G. Pecelli, Dichotomies for linear functional differential equations, J. Differential Equations 9 (1971) 555–579.

[13] K. Sakamoto, Estimates on the strength of exponential dichotomies and application of integral manifolds, J. Differential Equations 107 (1994) 259–279.

[14] A. Stokes, Local coordinates around a limit cycle of a functional differential equation with applications, J. Differential Equations 24 (1977) 153–172.

[15] M. Weedermann, Normal forms for neutral functional differential equations, in: Topics in Functional Differential and Difference Equations, Field Institute of Communication, Vol. 29, American Mathematical Society, Providence, RI, 2001.

Further reading

J.K. Hale, Solutions near simple periodic orbits of functional differential equations, J. Differential Equations 9 (1970) 126–183.

D. Henry, The adjoint of a linear functional differential equation and boundary value problems, J. Differential Equations 9 (1971) 55–66.

R. Nussbaum, Periodic solutions of analytic functional differential equations are analytic, Michigan Math. J. 20 (1973) 249–255.